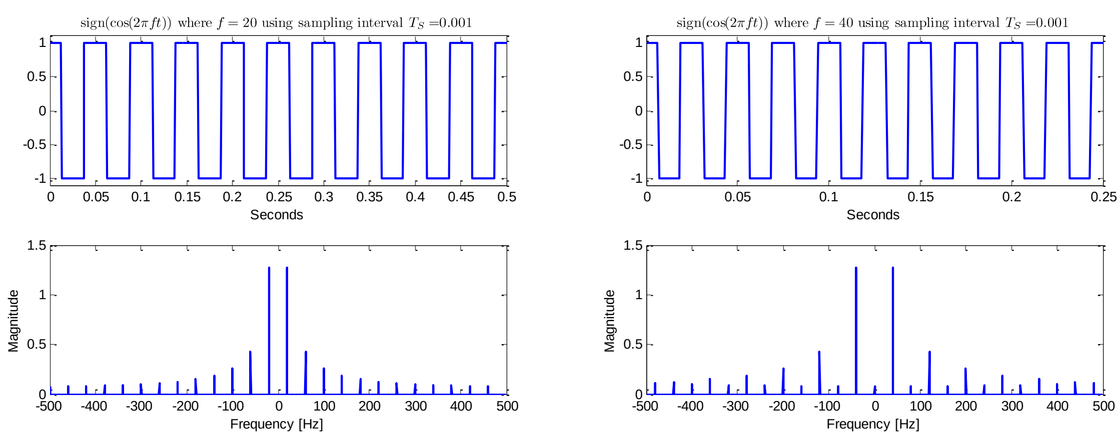
ECS 332: Solution for Problem Set 4

Problem 1: Aliasing and periodic square wave

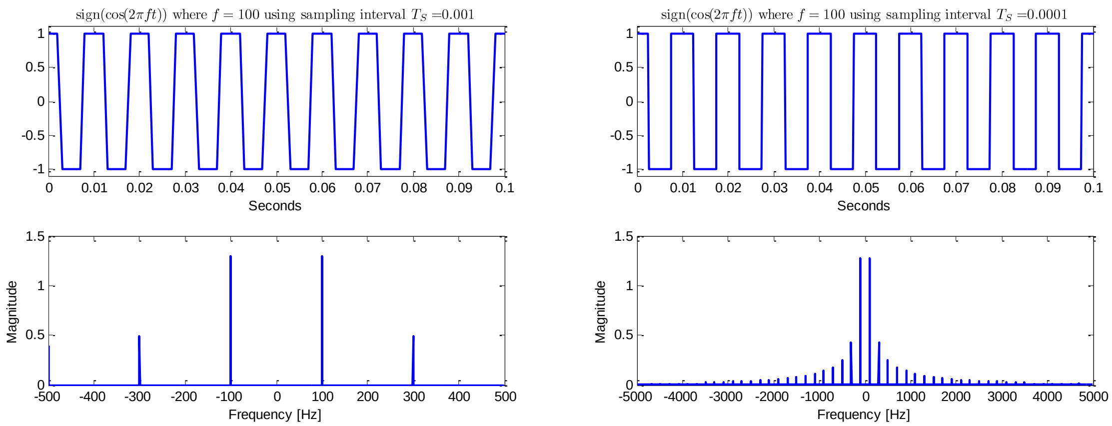
In the time domain, the switching between the values -1 and 1 should be faster as we increase f_0. In the frequency domain, we should get impulses (spikes) at all the odd-integer multiples of $\pm f_0$ Hz. The center spikes (at $\pm f_0$) should be the largest among them. All the plots below are adjusted so that they show 10 periods of the "original signal" in the time domain.

From the plots, as we increase f_0 from 10 to 20 Hz, the locations of spikes changes from all the odd-integer multiples of 10 Hz to all the odd-integer multiples of 20 Hz. In particular, we see the spikes at ±20, ±60, ±100, ±140, ±180, ±220, ±260, ±300, ±340, ±380, ±420, ±460. Note that plotspec only plots from $[-f_s/2, f_s/2)$. So, we see a spike at -500 but not 500. Of course, the Fourier transform of the sampled waveform is periodic and hence when we replicate the spectrum every f_s, we will have a spike at 500. Note that in reality, we should also see spikes at ±540, ±580, ±620, ±660, and so on. However, because the sampling rate is 1000, these high frequency spikes will suffer from aliasing and fold back into our viewing window $[-f_s/2, f_s/2)$. However, they fall back to the frequencies that already have spikes (for example, ±540 will fold back to ±460, and ±580 will fold back to ±420) and therefore the aliasing effect is not easily noticeable in the frequency domain.

When f_0 = 40, we start to see the aliasing effect in the frequency domain. Instead of seeing spikes only at ±40, ±120, ±200, ±280, ±360, ±440, the spikes at higher frequencies (such as ±520, ±600, and so on) fold back to lower frequencies (such as ±480, ±400, and so on). The plot still looks OK in the time domain.

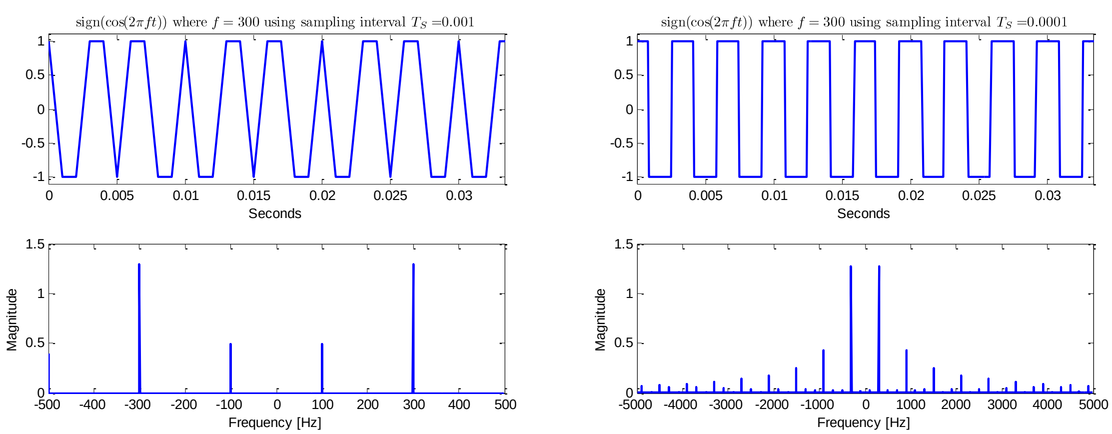


At high fundamental frequency f_0 = 100, we see stronger effect of aliasing. In the time domain, the waveform does not look quite “rectangular”. In the frequency domain, we only see the spikes at $\pm$100, $\pm$300, and 500. These are at the correct locations. However, there are too few of them to reconstruct a square waveform. The rest of the spikes are beyond our viewing window. We can’t see them directly because they fold back to the frequencies that are already occupied by the lower frequencies.



Our problem can be mitigated by reducing the sampling interval to T_S = 1/1e4 instead of T_S = 1/1e3. Now, the spikes show up again as shown by the plot on the right above.

Finally, at the highest frequency f_0 = 300, if we still use T = 1/1e3, the waveform will be heavily distorted in the time domain. This is shown in the left plot below. We have large spikes at $\pm$300 as expected. However, the next pair which should occur at $\pm$900 is out of the viewing window and therefore fold back to $\pm$100. Again, the aliasing effect can be mitigated by reducing the sampling time to T = 1/1e4 instead of T = 1/1e3. Now, more spikes show up at their expected places. Note that we can still see a lot of small spikes scattered across the frequency domain. These are again the spikes from higher frequency which fold back to our viewing window.



Q2 Nyquist sampling rate and Nyquist sampling interval

Sunday, July 17, 2011
2:09 PM

To apply the sampling theorem, we first need to find the value B where the signal in each part is bandlimited to.

Recall that $1[|t| \le a] \xrightarrow{\mathcal{F}} 2a \operatorname{sinc}(2\pi f a)$

By the duality theorem, we have

$$2a \operatorname{sinc}(2\pi a t) \xrightarrow{\mathcal{F}} 1[|-f| \le a]$$

Therefore, $\operatorname{sinc}(2\pi a t) \xrightarrow{\mathcal{F}} \frac{1}{2a} 1[|f| \le a]$

which implies that

$\operatorname{sinc}(2\pi f_0 t)$ is bandlimited to $B = f_0$

(a) $\operatorname{sinc}(100\pi t) = \operatorname{sinc}(2\pi \times 50 \times t) \Rightarrow B = 50$ Hz

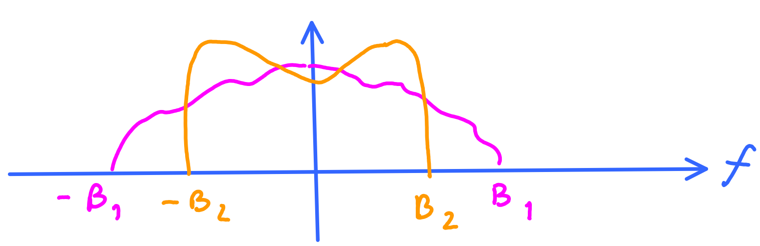
(b) Recall that for signals $g_1(t)$ bandlimited to B_1 and $g_2(t)$ " " B_2,

their product $g_1(t) g_2(t)$ is bandlimited to $B_1 + B_2$.

Hence, for $\operatorname{sinc}^2(100\pi t)$, $B = 50 + 50 = 100$ Hz

(c) Observe that for signals $g_1(t)$ bandlimited to B_1 and $g_2(t)$ " " B_2,

their linear combination $c_1 g_1(t) + c_2 g_2(t)$ is bandlimited to $\max\{B_1, B_2\}$.



So, for $\text{sinc}(100\pi t) + \text{sinc}(50\pi t)$, $B = \max\{50, 25\} = 50$ Hz

(d) Use the observation from parts (b) and (c).

For $\text{sinc}(100\pi t) + 3\,\text{sinc}^2(60\pi t)$,

$$B = \max\{50, 2\times 60\} = 120 \text{ Hz}.$$

(e) Use the same observation as in part (b).

For $\text{sinc}(50\pi t)\,\text{sinc}(100\pi t)$, $B = 25 + 50 = 75$ Hz.

The Nyquist sampling rate is $2 \times B$.

The Nyquist sampling interval is $\frac{1}{2B}$.

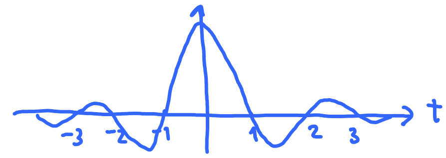
The table below summarizes the answers for this question:

	B	$R_{Nyquist}$ [sample/sec]	$T_{Nyquist}$ [sec]
(a)	50	100	0.01
(b)	100	200	0.005
(c)	50	100	0.01
(d)	60	120	1/120
(e)	75	150	1/150

Q3 Sinc Reconstruction of Sinc

Thursday, August 30, 2012
1:51 PM

The signal under consideration is $g(t) = \mathrm{sinc}(\pi t)$.



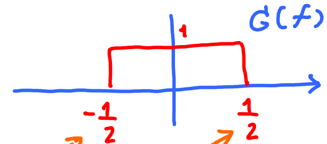
note that, in MATLAB, this function is implemented by sinc(t) because the built-in MATLAB sinc function has already included the π.

(a) The Fourier transform

$g(0) = 1$. So, $\int_{-\infty}^{\infty} G(f)\, df = 1$.

To have area = 1, the hight of the rectangular pulse must be 1 as well.

$$g(t) = \mathrm{sinc}(\pi t) = \frac{\sin(2\pi (\frac{1}{2}) t)}{\pi t} \xrightarrow{\mathcal{F}}$$

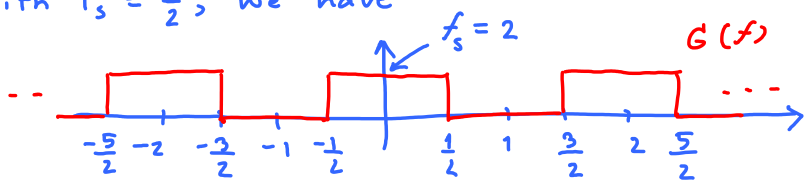


(b) The Nyquist sampling rate is given by $2 \times f_{max} = 2 \times \frac{1}{2} = 1$ samble/sec.

(c) In class, we have seen that

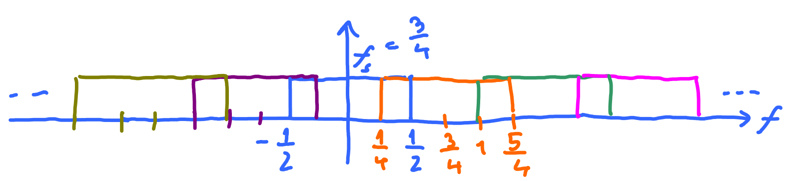
$$G_\delta(f) = f_s \sum_{n=-\infty}^{\infty} G(f - n f_s) \quad \text{where } f_s = \frac{1}{T_s}.$$

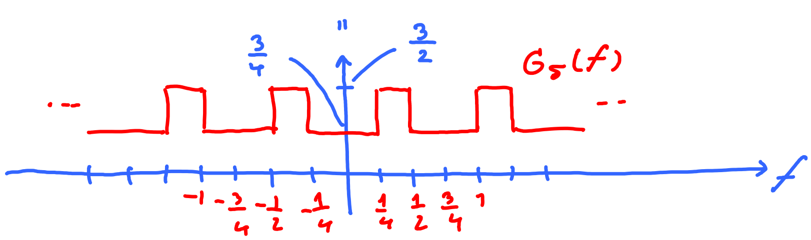
(c.i) With $T_s = \frac{1}{2}$, we have



Periodic with "period" = 2.

(c.ii) With $T_s = \frac{4}{3}$, we have





Periodic with "period" = $\frac{3}{4}$.

(d.i)

With $T_s = 1$, $g[n] = g(nT_s) = g(n\times 1) = g(n) = \text{sinc}(\pi n)$.

(d.i.i) From the plot of $\text{sinc}(\pi n)$ drawn earlier, we have

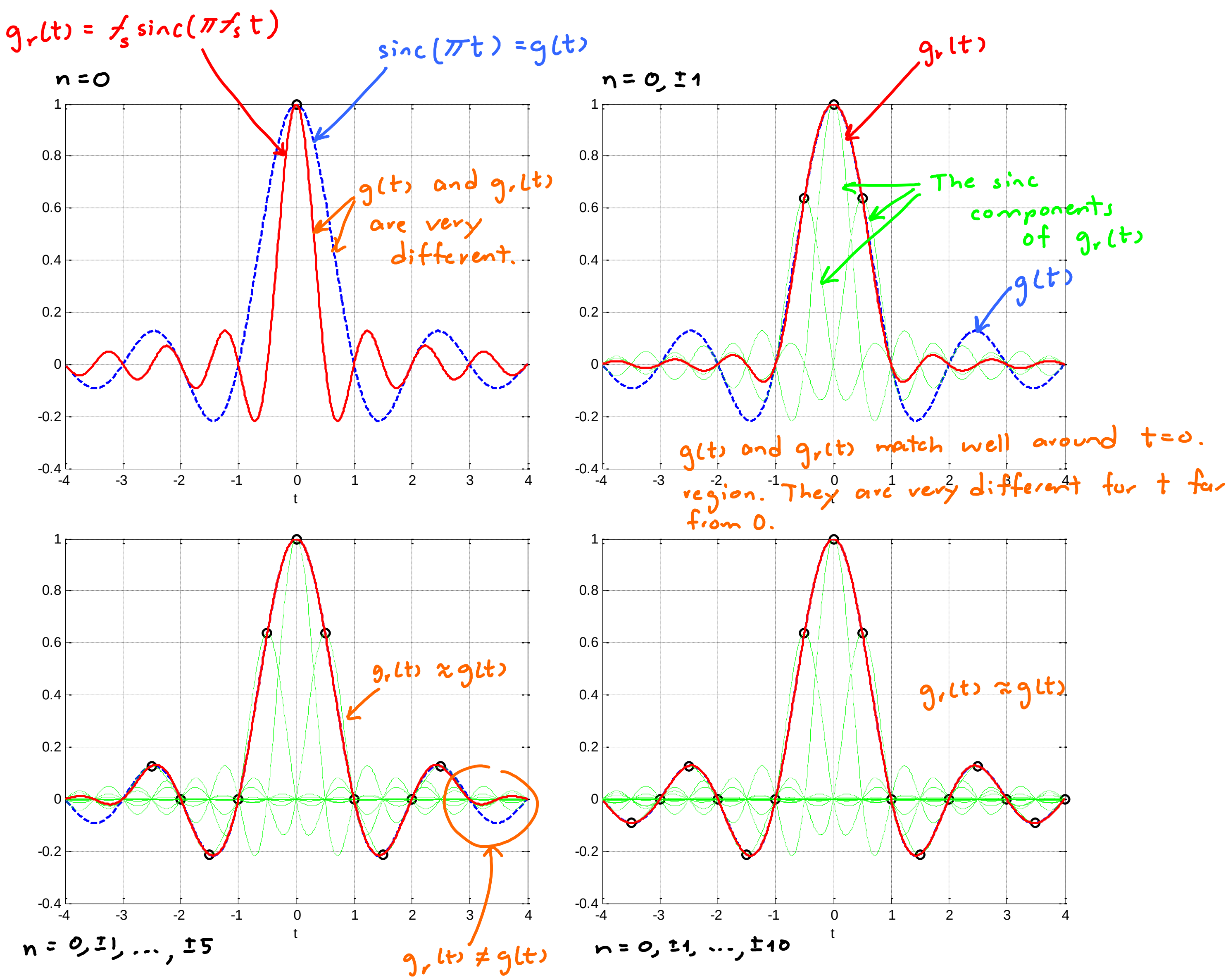
$$g[n] = \begin{cases} 1, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

(d.i.ii) Because $g[n] = 0$ when $n \neq 0$,

$$g_r(t) = g[0]\,\text{sinc}(\pi t) = 1 \times \text{sinc}(\pi t) = \text{sinc}(\pi t) = g(t)$$

Note that with $T_s = 1$, we have $f_s = 1$ which is the same as the Nyquist sampling rate. Therefore, we are on the border line of the successful reconstruction.

(d.ii) Reminder: MATLAB's sinc function is $\text{sinc}(x) = \sin(\pi x)/\pi x$, which is different from $\text{sinc}(x) = \sin(x)/x$ that we defined for our class. Therefore, when we use MATLAB to plot $\text{sinc}(\pi t)$, we do not put the "π" in the formula. MATLAB will automatically insert the "π" for us.



Observation: As we increase the number of terms in the summation, $g(t)$ is better approximated by $g_r(t)$.